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Entropy and information gain in quantum continual measurements

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1 Introduction

The theory of measurements continuous in time in quantum mechanics (quantum continual measurements) has been formulated by using the notions of instrument, positive operator valued (POV) measure, etc. [1, 2], by using quantum stochastic differential equations [3, 4] and by using classical stochastic differential equations (SDE's) for vectors in Hilbert spaces or for trace-class operators [5, 6, 7, 8]. In the same times Ozawa made developments in the theory of instruments [9, 10] and introduced the related notions of *a posteriori* states [11] and of information gain [12].

In Section 2 we introduce a simple class of SDE's relevant to the theory of continual measurements and we recall how they are related to instruments and *a posteriori* states and, so, to the general formulation of quantum mechanics [13]. In Section 3 we shall introduce and use the notion of information gain and the other results of paper [12] inside the theory of continual measurements.

2 Stochastic differential equations and instruments

Let $\mathcal{H}$ be a separable complex Hilbert space, associated to the quantum system of interest. Let us denote by $\mathcal{B}(\mathcal{H})$ the space of bounded linear operators on $\mathcal{H}$ and by $\mathcal{T}(\mathcal{H})$ the trace-class on $\mathcal{H}$, i.e. $\mathcal{T}(\mathcal{H}) = \big\{\rho \in \mathcal{B}(\mathcal{H}) : \|\rho\| \equiv \operatorname{Tr}\{\sqrt{\rho^*\rho}\} < \infty\big\}$. Let $\mathcal{S}(\mathcal{H}) \subset \mathcal{T}(\mathcal{H})$ be the set of all statistical operators (*states*) on $\mathcal{H}$. Commutators and anticommutators are denoted by $[\ ,\ ]$ and $\{\ ,\ \}$, respectively.

Let H, L_j, S_h, $j, h = 1, 2, \ldots$, be bounded operators on $\mathcal{H}$ such that $H = H^\dagger$, $\sum_{j=1}^{\infty} L_j^\dagger L_j$ and $\sum_{h=1}^{\infty} S_h^\dagger S_h$ are strongly convergent in $\mathcal{B}(\mathcal{H})$. Let J_k be a bounded linear map on $\mathcal{T}(\mathcal{H})$ such that its adjoint J_k^* is a normal, completely positive map on $\mathcal{B}(\mathcal{H})$ and $\sum_{k=1}^{\infty} J_k^*[\mathbb{1}]$ is strongly convergent to a bounded

operator. Then, we introduce the following operators on $\mathcal{T}(\mathcal{H})$:

$$
\begin{aligned}
\mathcal{L}_0[\rho] &= -\mathrm{i}[H,\rho] + \sum_{j=1}^{\infty}\Big(L_j\rho L_j^\dagger - \frac{1}{2}\Big\{L_j^\dagger L_j,\,\rho\Big\}\Big) \\
&\quad + \sum_{k=1}^{\infty}\Big(J_k[\rho] - \frac{1}{2}\left\{J_k^*[\mathbb{1}],\,\rho\right\}\Big), && (1) \\
\mathcal{L}_1[\rho] &= \sum_{h=1}^{\infty}\Big(S_h\rho S_h^\dagger - \frac{1}{2}\Big\{S_h^\dagger S_h,\,\rho\Big\}\Big), && (2) \\
\mathcal{L} &= \mathcal{L}_0 + \mathcal{L}_1. && (3)
\end{aligned}
$$

The adjoint operators of $\mathcal{L}$, $\mathcal{L}_0$, $\mathcal{L}_1$ are generators of norm-continuous quantum dynamical semigroups [14, 15].

Let us now consider the following linear SDE (in the sense of Itô) for trace-class operators:

$$
\begin{aligned}
\mathrm{d}\sigma_t &= \mathcal{L}[\sigma_{t^-}]\,\mathrm{d}t + \sum_{j=1}^{\infty}\Big(\widetilde{L}_j(t)\sigma_{t^-} + \sigma_{t^-}\widetilde{L}_j(t)^\dagger\Big)\,\mathrm{d}\widetilde{W}_j(t) + \\
&\quad + \sum_{k=1}^{\infty}\left(\frac{1}{\lambda_k}\,J_k[\sigma_{t^-}] - \sigma_{t^-}\right)\big(\mathrm{d}N_k(t) - \lambda_k\,\mathrm{d}t\big); && (4)
\end{aligned}
$$

the initial condition is $\sigma_0 = \rho \in \mathcal{S}(\mathcal{H})$ (a non-random state) and we have set

$$
\widetilde{L}_j(t) = \mathrm{e}^{\mathrm{i}\omega_j t} L_j\,, \qquad\qquad \omega_j \in \mathbb{R}\,. \tag{5}
$$

The processes $\widetilde{W}_j(t)$ are independent standard Wiener processes, the $N_k(t)$ are independent Poisson processes of intensity $\lambda_k > 0$, which are also independent of the Wiener processes; we assume $\sum_k \lambda_k < +\infty$.

These processes are realized in a probability space $(\Omega, \mathcal{F}, Q)$; the sample space Ω is, roughly speaking, the set of possible trajectories for the processes $\widetilde{W}_j$, N_k, the event space $\mathcal{F}$ is the σ-algebra of sets of trajectories to which a probability can be given and Q is the probability law under which $\widetilde{W}_j$, N_k are independent Wiener and Poisson processes. Moreover, let $\mathcal{F}_t$ be the collection of events which are specified by giving conditions involving times only in the interval $[0,t]$. We also ask $\mathcal{F} = \mathcal{F}_\infty$. In mathematical terms the $\widetilde{W}_j$, N_k are canonical Wiener and Poisson processes, $\{\mathcal{F}_t,\, t \geq 0\}$ is their natural filtration and $\mathcal{F} = \bigvee_{t\geq 0} \mathcal{F}_t$. Finally, let us denote by $\mathbb{E}_Q$ the expectation with respect to the probability Q, i.e. $\mathbb{E}_Q[A] = \int_\Omega A(\omega)Q(\mathrm{d}\omega)$.

For every $F \in \mathcal{F}_t$ and every initial condition $\rho \in \mathcal{S}(\mathcal{H})$, let us set

$$
\mathcal{I}_t(F)[\rho] = \mathbb{E}_Q[1_F\sigma_t] \equiv \int_F \sigma_t(\omega)Q(\mathrm{d}\omega); \tag{6}
$$

1_F is the indicator function of the set F, i.e. $1_F(\omega) = 1$ if $\omega \in F$ and $1_F(\omega) = 0$ if $\omega \notin F$. The map $\mathcal{I}_t$ turns out to be a (completely positive) instrument [9]

with value space $(\Omega, \mathcal{F}_t)$ and $\mathcal{I}_t(\cdot)^*[\mathbb{1}]$ is the associated POV measure. Then we set, $\forall F \in \mathcal{F}_t$,

$$P_\rho(F) = \operatorname{Tr}\left\{\mathcal{I}_t(F)^*[\mathbb{1}]\rho\right\} = \mathbb{E}_Q\left[\|\sigma_t\|\, 1_F\right]. \tag{7}$$

The important point in this formula is that $\|\sigma_t\|$ is a Q-martingale and this implies that the time dependent probability measures on the r.h.s. are consistent and define a unique probability P_ρ on $(\Omega, \mathcal{F})$.

The interpretation of eqs. (6) and (7) is that $\{\mathcal{I}_t,\, t \geq 0\}$ is the family of instruments describing the continual measurement, the processes $\widetilde{W}_j$, N_k represent the output of this measurement and P_ρ is the physical probability law of the output.

From eq. (6) it follows that

$$\eta_t = \mathcal{I}_t(\Omega)[\rho] = \mathbb{E}_Q[\sigma_t] \tag{8}$$

is the state to be attributed to the system at time t if the output of the measurement is not taken into account or not known; it can be called the *a priori* state at time t. It turns out that the *a priori* states satisfy the master equation

$$\frac{\mathrm{d}}{\mathrm{d}t}\,\eta_t = \mathcal{L}[\eta_t]\,, \qquad \eta_0 = \rho\,. \tag{9}$$

If we introduce the random states

$$\rho_t = \frac{\sigma_t}{\|\sigma_t\|}\,, \tag{10}$$

then we have, $\forall F \in \mathcal{F}_t$,

$$\mathcal{I}_t(F)[\rho] = \mathbb{E}_Q[1_F \sigma_t] = \mathbb{E}_{P_\rho}\left[1_F\, \frac{\sigma_t}{\|\sigma_t\|}\right] = \int_F \rho_t(\omega) P_\rho(\mathrm{d}\omega). \tag{11}$$

According to [11], $\rho_t(\omega)$ is a family of *a posteriori* states for the instrument $\mathcal{I}_t$ and the initial state ρ, i.e. $\rho_t(\omega)$ is the state to be attributed to the system at time t when the trajectory ω of the output is known, up to time t. Note that $\eta_t = \mathbb{E}_Q[\sigma_t] = \mathbb{E}_{P_\rho}[\rho_t]$.

By using Itô's calculus, we find that the *a posteriori* states satisfy the non-linear SDE

$$\begin{aligned}\mathrm{d}\rho_t \quad &= \quad \mathcal{L}\left[\rho_{t^-}\right]\mathrm{d}t + \sum_{j=1}^{\infty}\left[\widetilde{L}_j(t)\rho_{t^-} + \rho_{t^-}\widetilde{L}_j(t)^\dagger - m_j(t)\rho_{t^-}\right]\mathrm{d}W_j(t) + \\ &\quad + \sum_{k=1}^{\infty}\left[\frac{1}{\nu_k(t)}\,J_k[\rho_{t^-}] - \rho_{t^-}\right]\big(\mathrm{d}N_k(t) - \nu_k(t)\,\mathrm{d}t\big),\end{aligned} \tag{12}$$

where

$$W_j(t) = \widetilde{W}_j(t) - \int_0^t m_j(s)\,\mathrm{d}s\,, \tag{13}$$

$$m_j(t) = \operatorname{Tr}\left\{\rho_{t^-}\left(\widetilde{L}_j(t) + \widetilde{L}_j(t)^\dagger\right)\right\}, \qquad \nu_k(t) = \operatorname{Tr}\left\{\rho_{t^-}J_k^*[\mathbb{1}]\right\}. \tag{14}$$

Under the physical probability law P_ρ, the processes $W_j(t)$ are independent standard Wiener processes and the $N_k(t)$ are counting processes with stochastic intensity $\nu_k(t)$. In eq. (12) the sum in the jump term is only on the set where the stochastic intensity $\nu_k(t)$ is different from zero.

Formulae for the moments of the output can be obtained by the technique of the characteristic operator [2, 3, 4]. Let $h_{k\alpha}$ be real test functions in a suitable space; we define the characteristic operator $\mathcal{G}$ by

$$\begin{aligned} \mathcal{G}_t(h)[\rho] \quad &= \quad \mathbb{E}_{P_\rho}\Bigg[\exp\Bigg\{\mathrm{i}\sum_j \int_0^t h_{j1}(s)\,\mathrm{d}\widetilde{W}_j(s) \\ &\qquad + \mathrm{i}\sum_k \int_0^t h_{k2}(s)\,\mathrm{d}N_k(s)\Bigg\}\,\rho_t\Bigg]\,; \end{aligned} \tag{15}$$

then, $\mathrm{Tr}\left\{\mathcal{G}_t(h)[\rho]\right\}$ is the characteristic functional of the output up to time t (the Fourier transform of P_ρ restricted to $\mathcal{F}_t$). By Itô's calculus we obtain

$$\frac{\mathrm{d}}{\mathrm{d}t}\,\mathcal{G}_t(h)[\rho] = \mathcal{K}_t(h)\circ\mathcal{G}_t(h)[\rho]\,, \tag{16}$$

$$\begin{aligned} \mathcal{K}_t(h)[\rho] \quad &= \quad \mathcal{L}[\rho] + \mathrm{i}\sum_j h_{j1}(t)\left[\widetilde{L}_j(t)\rho + \rho\widetilde{L}_j(t)^\dagger\right] \\ &\qquad -\frac{1}{2}\sum_j h_{j1}(t)^2\rho + \sum_k \left\{\exp\left[\mathrm{i}h_{k2}(t)\right] - 1\right\} J_k[\rho]\,. \end{aligned} \tag{17}$$

All the moments can be obtained by functional differentiation of the characteristic functional. In particular, the mean values are expressed in terms of the *a priori* states as

$$\mathbb{E}_{P_\rho}\left[\widetilde{W}_j(t)\right] = \int_0^t \mathbb{E}_{P_\rho}\left[m_j(s)\right]\mathrm{d}s\,, \quad \mathbb{E}_{P_\rho}\left[N_k(t)\right] = \int_0^t \mathbb{E}_{P_\rho}\left[\nu_k(s)\right]\mathrm{d}s\,,$$

$$\mathbb{E}_{P_\rho}\left[m_j(s)\right] = \mathrm{Tr}\left\{\eta_s\left(\widetilde{L}_j(s) + \widetilde{L}_j(s)^\dagger\right)\right\}\,, \quad \mathbb{E}_{P_\rho}\left[\nu_k(s)\right] = \mathrm{Tr}\left\{J_k[\eta_s]\right\}\,,$$

and the second moments are given by

$$\begin{aligned} &\mathbb{E}_{P_\rho}\big[X_{j\alpha}(t)X_{i\beta}(s)\big] = \delta_{ij}\delta_{\alpha\beta}\int_0^{\min\{t,s\}} \mathrm{d}\tau\,\left(\delta_{\alpha 1} + \delta_{\alpha 2}\,\mathrm{Tr}\left\{J_i[\eta_\tau]\right\}\right) \\ &+ \int_0^t \mathrm{d}\tau_1 \int_0^{\min\{s,\tau_1\}} \mathrm{d}\tau_2\,\mathrm{Tr}\left\{\mathcal{A}_{j\alpha}(\tau_1)\circ\mathrm{e}^{\mathcal{L}(\tau_1-\tau_2)}\circ\mathcal{A}_{i\beta}(\tau_2)[\eta_{\tau_2}]\right\} \\ &+ \int_0^s \mathrm{d}\tau_2 \int_0^{\min\{t,\tau_2\}} \mathrm{d}\tau_1\,\mathrm{Tr}\left\{\mathcal{A}_{i\beta}(\tau_2)\circ\mathrm{e}^{\mathcal{L}(\tau_2-\tau_1)}\circ\mathcal{A}_{j\alpha}(\tau_1)[\eta_{\tau_1}]\right\}, \end{aligned}$$

where $X_{j1}(t) = \widetilde{W}_j(t)$, $X_{j2}(t) = N_j(t)$, $\mathcal{A}_{j1}(t)[\rho] = \widetilde{L}_j(t)\rho + \rho\widetilde{L}_j(t)^\dagger$, $\mathcal{A}_{j2}(t) = J_j$.

The class of SDE's presented here is a particular case of the one studied in [16] and, while not so general, it contains the main detection schemes found in quantum optics [17]; also the chosen time-dependence is natural for some systems typical of quantum optics under the so called heterodyne/homodyne detection scheme.

3 Entropy and information gain

In [12] a measurement is called quasi-complete if the *a posteriori* states are pure for every pure initial state and it is called complete if the *a posteriori* states are pure for every (pure or mixed) initial state. So, we call *quasi-complete* the continual measurement of Section 2 if the *a posteriori* states ρ_t are *pure* (P_ρ-almost surely) *for all* t *and for all pure initial conditions* ρ. In [18] we proved that

Theorem 1 *The continual measurement of Section 2 is quasi-complete if and only if* $\mathcal{L}_1 = 0$ *and* $\dfrac{J_k[\rho]}{\operatorname{Tr}\{J_k[\rho]\}}$ *is a pure state for every* k *and for every pure state* ρ. *In this case there exists a partition* A_1, A_2 *of the integer numbers such that for some* $R_k \in \mathcal{B}(\mathcal{H})$ *and for some monodimensional projection* P_k *we can write* $J_k[\rho] = R_k \rho R_k^\dagger$, *for* $k \in A_1$, $J_k[\rho] = \operatorname{Tr}\{\rho J_k^*[\mathbb{1}]\} P_k$, *for* $k \in A_2$.

Our continual measurement can not be complete in the sense of [12] for a fixed time; however, it can be "asymptotically complete". Examples of this behaviour in the case of linear systems are given in [19]. In [18], we proved that

Theorem 2 *Let the continual measurement of Section 2 be quasi-complete and let* $\mathcal{H}$ *be finite-dimensional. If for every time* t *it does not exist a bidimensional projection* P_t *such that,* $\forall j,k$, $P_t\left(\widetilde{L}_j(t) + \widetilde{L}_j(t)^\dagger\right)P_t = z_j(t)P_t$, $P_t J_k^*[\mathbb{1}]P_t = q_k(t)P_t$ *for some complex numbers* $z_j(t)$ *and* $q_k(t)$, *then eq. (12) maps asymptotically, for* $t \to \infty$, *mixed states into pure ones, in the sense that for every initial condition* ρ *we have* P_ρ*-almost surely* $\lim_{t\to\infty} \operatorname{Tr}\{\rho_t(\mathbb{1} - \rho_t)\} = 0$.

The proof of the theorems above is based on the study of the *a posteriori linear entropy* (or *purity*) $\operatorname{Tr}\{\rho_t(\mathbb{1} - \rho_t)\}$ and of its mean value. However, physically more interesting quantities are the von Neumann entropy and the relative entropy: for $x, y \in \mathcal{S}(\mathcal{H})$, $S[x] = -\operatorname{Tr}\{x \ln x\} \geq 0$, $S[x|y] = \operatorname{Tr}\{x \ln x - x \ln y\} \geq 0$ (they can also diverge) [15]. In our case we have the initial state $\rho = \rho_0 = \sigma_0 = \eta_0$ and the *initial entropy* $S[\rho]$, the *a priori* state η_t and the *a priori entropy* $S[\eta_t]$, the *a posteriori* states ρ_t and the *mean a posteriori entropy*

$$\mathbb{E}_{P_\rho}\big[S[\rho_t]\big] = \mathbb{E}_Q\big[\|\sigma_t\| \ln \|\sigma_t\| - \operatorname{Tr}\{\sigma_t \ln \sigma_t\}\big]\,. \tag{18}$$

By some direct computations, we obtain a first relation among these quantities:

$$S[\eta_t] - \mathbb{E}_{P_\rho}\big[S[\rho_t]\big] = \mathbb{E}_{P_\rho}\big[S[\rho_t|\eta_t]\big] \geq 0\,. \tag{19}$$

Following [12], we can also introduce the *amount of information* of the continual measurement

$$I[\rho;t] = S[\rho] - \mathbb{E}_{P_\rho}\big[S[\rho_t]\big] \tag{20}$$

and the classical amount of information. To introduce this last quantity we need some notations. Let us set $P_\rho(\mathrm{d}\omega;t) = \|\sigma_t(\omega)\| Q(\mathrm{d}\omega)$, let $\rho = \sum_\alpha w_\alpha \rho_\alpha$ be the orthogonal decomposition of ρ into pure states and P_{ρ_α}, σ_t^α, ρ_t^α, η_t^α, $m_j^\alpha(t)$, $\nu_k^\alpha(t)$ be defined starting from ρ_α as P_ρ, σ_t, ρ_t, η_t, $m_j(t)$, $\nu_k(t)$ are defined starting from ρ. Then, the *classical amount of information* of the continual measurement is defined by

$$\begin{aligned}
\text{c-}I[\rho;t] &= \sum_\alpha w_\alpha \int_\Omega \ln\left(\frac{P_{\rho_\alpha}(\mathrm{d}\omega;t)}{P_\rho(\mathrm{d}\omega;t)}\right) P_{\rho_\alpha}(\mathrm{d}\omega;t) \\
&= \sum_\alpha w_\alpha \mathbb{E}_{P_{\rho_\alpha}}\left[\ln\frac{\|\sigma_t^\alpha\|}{\|\sigma_t\|}\right] \\
&= \mathbb{E}_Q\left[\sum_\alpha w_\alpha \|\sigma_t^\alpha\| \ln\|\sigma_t^\alpha\| - \|\sigma_t\|\ln\|\sigma_t\|\right].
\end{aligned} \tag{21}$$

By classical arguments, c-$I[\rho;t]$ is always positive [12]: c-$I[\rho;t] \geq 0$, $\forall t \geq 0$, $\forall \rho \in \mathcal{S}(\mathcal{H})$. Obviously, we have $I[\rho;t] \leq S[\rho]$, $I[\rho;0] = 0$, c-$I[\rho;0] = 0$. If it exists an equilibrium state η_{eq} ($\mathcal{L}[\eta_{\mathrm{eq}}] = 0$), by (19) we have also $I[\eta_{\mathrm{eq}};t] \geq 0$.

Theorem 3 *The classical amount of information of the continual measurement of Section 2 is non-decreasing in time and*

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}\,\text{c-}I[\rho;t] &= \sum_\alpha w_\alpha \mathbb{E}_{P_{\rho_\alpha}}\left[\frac{1}{2}\sum_j m_j^\alpha(t)^2 + \sum_k \nu_k^\alpha(t)\ln\nu_k^\alpha(t)\right] \\
&- \mathbb{E}_{P_\rho}\left[\frac{1}{2}\sum_j m_j(t)^2 + \sum_k \nu_k(t)\ln\nu_k(t)\right] \\
&= \sum_\alpha w_\alpha \mathbb{E}_{P_{\rho_\alpha}}\left[\frac{1}{2}\sum_j \big(m_j^\alpha(t) - m_j(t)\big)^2\right. \\
&+ \left.\sum_k \nu_k(t)\left(1 - \frac{\nu_k^\alpha(t)}{\nu_k(t)} + \frac{\nu_k^\alpha(t)}{\nu_k(t)}\ln\frac{\nu_k^\alpha(t)}{\nu_k(t)}\right)\right] \geq 0\,.
\end{aligned} \tag{22}$$

To prove this theorem one has to differentiate the last expression in (21) and to use the relationships among Q, P_ρ, P_{ρ_α}.

For quasi-complete measurements the information gain $I[\rho;t]$ has a nice behaviour.

Theorem 4 *The continual measurement of Section 2 is quasi-complete if and only if the amount of information $I[\rho;t]$ is non-negative for any $\rho \in \mathcal{S}(\mathcal{H})$ with $S[\rho] < +\infty$ and any $t \geq 0$. Moreover, if it is quasi-complete, we have*

$I[\rho;t] \geq$ c-$I[\rho;t] \geq 0$, $I[\rho;t] \geq I[\rho;s]$ for any t, any $s < t$ and any state ρ with $S[\rho] < +\infty$.

Proof. All the statements but the last one are a particularization of Theorems 1 and 2 of [12] to our case. The last statement needs the use of conditional expectations. We have $I[\rho;t] - I[\rho;s] = \mathbb{E}_{P_\rho}\big[S[\rho_s] - \mathbb{E}_{P_\rho}[S[\rho_t]|\mathcal{F}_s]\big]$; by (12) $S[\rho_s] - \mathbb{E}_{P_\rho}[S[\rho_t]|\mathcal{F}_s]$ is the amount of information at time t when the initial time is s and the initial state is ρ_s and, so, it is non-negative for a quasi-complete measurement. □

Finally, if $\mathcal{H}$ is finite-dimensional, the vanishing of the purity implies the vanishing of the entropy; therefore, we have the asymptotic completeness also in the sense of the vanishing of the entropy:

> *The hypotheses of Theorem 2 imply also that $\lim_{t\to+\infty} S[\rho_t] = 0$, P_ρ-almost surely, and $\lim_{t\to+\infty} I[\rho;t] = S[\rho]$.*

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